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***Abstract*— We provide a reduction model approach for achieving global exponential stabilization of linear systems with a time-varying pointwise delay in the input. We allow the delay to be discontinuous and uncertain. We also provide a stability result based on a different dynamic extension that ensures input-to-state stability with respect to additive uncertainties on the dynamics. Instead of the usual Lyapunov-Krasovskii or Razumikhin methods, we use a trajectory based approach.**

Key Words: **delay, time-varying, stabilization**

I. Introduction

This note continues our search (begun in [24], [25], and [26]) for novel methods to prove global stabilization for systems with time delays. Our search is motivated by the ubiquity of input delays in engineering applications with feedback [8], [9], [13], [14], [23], [28], [31], [32] and the fact that classical methods for undelayed systems can rarely cope with the more complicated systems that result from allowing input delays [33]. For instance, while classical Lyapunov functions are suited for proving stability of systems without delays, one often replaces Lyapunov functions by Lyapunov-Krasovskii functionals [11] or Ruzumikhin functions to cope with stability problems for delayed systems.

It is often useful to prove stabilizability of time delay systems using the following two step process. First, one solves the stabilization problem with the input delays set to zero, often by building a Lyapunov function for the closed loop undelayed system and then finding decay estimates for the Lyapunov function. Then, one reintroduces the input delay and converts the Lyapunov function into a Lyapunov-Krasovskii functional for the corresponding input delayed systems, to find upper bounds on the input delays that the system can tolerate without destroying the stability. One advantage of this so-called emulation approach is that it can allow the use of relatively simple controllers. However, emulation cannot always cope with communications and other engineering applications that can have long delays [12].

Reduction is a useful alternative to emulation, where the control uses a dynamic extension and can compensate for arbitrarily long input delays [1], [5], [10], [21], [29]. It has its origins in the classical Smith predictor [34] for linear systems and so is also called prediction [17], [20], but recent prediction results apply to a much wider class of systems, including adaptive and perturbed systems [2], [3], [4], [6], [19]. Recently, Bresch-Pietri and Petit used a transport PDE, reduction, and a generalization of an inequality due to Halanay [15], [16] to prove stability of linear time invariant systems with known input delays $h(t)$ that can satisfy $\dot{h}(t) > 1$ for some t's [7]. This differs from the usual treatments that assume that $\sup_t \dot{h}(t) \leq 1$. Thus, [7] is a significant advance in the area of stabilization of systems with time-varying delays. A key assumption in [7] is that $|\dot{h}|$ is small on average, without requiring a bound on $|\dot{h}|$, so [7] covers chattering phenomena in delays that occur in many engineering systems and so are of considerable interest.

Here, we pursue a related line of research involving reduction, but our results differ from [7] in several important ways. First, we allow discontinuities, as well uncertainties, in the delays. Second, our assumptions are different, and they lead to stability proofs that are based on our trajectory approach from [24] instead of transport PDEs. The approach from [24] was not used in [7]. Finally, we provide an alternative approach that is based on a dynamical extension from [22], [31] from the theory of spectrum assignment.

In the next section, we provide preliminaries, including a key generalization of Halanay's inequality. In Section III, we present our main result, which allows uncertain or discontinuous delays. In Section IV, we provide our alternative approach using dynamic extensions from [31]. In Section V, we illustrate our work using an unstable second-order dynamics. We close in Section VI by summarizing the value added by our work and suggesting future research problems. This paper is a companion to [27], which uses our trajectory based approach from [24] without using reduction, under completely different assumptions from the ones in this paper and assuming the delays are known.

II. Preliminaries

In all of what follows, all dimensions are arbitrary. The standard Euclidean norm of vectors, and the induced matrix norm, are denoted by $|\cdot|$, I_r is the identity matrix in dimension r, and $|\cdot|_{\mathcal{I}}$ denotes the supremum over any interval $\mathcal{I} \subseteq \mathbb{R}$. Also, $\lambda_{\max}(Q) > 0$ (resp., $\lambda_{\min}(Q) > 0$) denotes the largest (resp., smallest) eigenvalue of any symmetric positive definite matrix Q. Let C^1 be the set of all continuously differentiable functions, where the domains and ranges will be clear from the context. For any constant $\tau > 0$, let $C([-\tau, 0], \mathbb{R}^n)$ be the set of all continuous $\mathbb{R}^n$-valued functions having the domain $[-\tau, 0]$. We abbreviate this set as $C_{\rm in}$, and call it the set of all *initial functions*. For what follows, we take $\tau = \sup_t h(t)$, where $h(t)$ is

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the delay. For any continuous function $\varphi : [-\tau, \infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define φ_t by $\varphi_t(m) = \varphi(t + m)$ for all $m \in [-\tau, 0]$. A function defined on an interval $\mathcal{I} \subseteq \mathbb{R}$ is called *piecewise continuous* provided it is continuous except at finitely many points on each bounded subinterval of $\mathcal{I}$. We use this variant of the Halanany inequality:

Lemma 1: Let $X : [0, \infty) \to [0, \infty)$ be a piecewise C^1 function that admits constants $g \geq 0$ and $a_s \geq 0$ and piecewise continuous functions $a : [0, \infty) \to [-a_s, \infty)$, $b : [0, \infty) \to [0, \infty)$, and $\lambda : [0, \infty) \to [0, \infty)$ such that

$$\dot{X}(t) \leq -a(t)X(t) + b(t)\sup_{s\in[t-g,t]} X(s) + \lambda(t) \qquad (1)$$

holds for all $t \geq g$. Assume that there exist two constants $T > 0$ and $\delta \in (0, 1)$ such that

$$e^{-\int_{t-T}^{t} a(\ell)\mathrm{d}\ell} + \int_{t-T}^{t} b(q)e^{-\int_{q}^{t} a(\ell)\mathrm{d}\ell}\mathrm{d}q \leq \delta \qquad (2)$$

holds for all $t \geq T + g$. Then, the inequality

$$\begin{array}{l} X(t) \leq \\ |X|_{[0,T+g]}\exp\left(\frac{\ln(\delta)}{T+g}(t - T - g)\right) + \frac{Te^{Ta_s}|\lambda|_{[g,t]}}{(1-\delta)^2} \end{array} \qquad (3)$$

holds for all $t \geq T + g$. □

For the proof of Lemma 1, see [27]; its proof is based on the trajectory approach from [24]. Estimate (3) is a special case of input-to-state stability, which we define next.

We first let $\mathcal{K}$ be the set of all strictly increasing continuous functions $\alpha : [0, \infty) \to [0, \infty)$ such that $\alpha(0) = 0$; if, in addition, α is unbounded, then we say that it is of class $\mathcal{K}_\infty$. We say that a continuous function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ provided for each $s \geq 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$. We say that a system of the form $\dot{x}(t) = f(x(t), u(t - h(t)), \varepsilon(t))$ having a controller u (with a time delay $h(t)$ that admits a constant $g > 0$ such that $0 \leq h(t) \leq g$ for all $t \geq 0$) is input-to-state stable (ISS) [18] with respect to the set of all piecewise continuous functions $\varepsilon : [0, \infty) \to \mathbb{R}^m$ provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t)| \leq \beta(|x_0|_{[-g,0]}, t) + \gamma(|\varepsilon|_{[0,t]}) \qquad (4)$$

holds for all $t \geq 0$, all continuous initial functions $x_0 : [-g, 0] \to \mathbb{R}^n$, and all piecewise continuous functions $\varepsilon : [0, \infty) \to \mathbb{R}^m$. This agrees with the more familiar global exponential stability condition when the perturbations ε are identically zero and $\beta(s, t) = c_1 se^{-c_2 t}$ for some constants $c_1 > 0$ and $c_2 > 0$.

III. Main Result

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t - h(t) - \gamma(t)) \qquad (5)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant, u is the control, h is a known delay, and γ represents uncertainty in the delay. We make several assumptions, but see below for results under different controllers that also provide input-to-state stability and which have several degrees of freedom in terms of parameters we can tune. First, we assume:

Assumption 1: The function h is of class C^1 and there is a constant $g > 0$ such that

$$0 \leq h(t) \leq g \qquad (6)$$

holds for all $t \geq 0$. The unknown function γ is piecewise continuous, and there are a known constant $\gamma_c \geq 0$ and a known continuous function γ_b such that the inequalities

$$0 \leq \gamma(t) \leq \gamma_b(t) \leq \gamma_c \qquad (7)$$

hold for all $t \geq 0$. □

Assumption 2: The pair (A, B) is controllable. □

By Assumption 2, there is a matrix $K \in \mathbb{R}^{m\times n}$ such that $R = A + BK$ is Hurwitz. This provides a constant $c > 0$ and a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QR + R^\top Q \leq -cQ\ . \qquad (8)$$

Choose a symmetric positive definite matrix $S \in \mathbb{R}^{n\times n}$ such that $Q = SS$, i.e., $S = \sqrt{Q}$. We also define

$$N = SA,\ M = SAS^{-1},\ B_K = BK, \qquad (9)$$

$$\alpha(t) = c - 2|M||\dot{h}(t)|\ ,\ \text{and} \qquad (10)$$

$$\begin{aligned} \beta(t) \ &= \ 2|S||B_K|e^{|A|h(t)}\Big(2|S^{-1}||\dot{h}(t)| + \gamma_b(t)a_1 \\ &\quad + a_2\int_{t-h(t)-\gamma_b(t)}^{t-h(t)} |\dot{h}(\ell)|\mathrm{d}\ell\Big) \end{aligned} \qquad (11)$$

where

$$\begin{array}{l} a_1 = |R||S^{-1}| + 2e^{|A|g}|B_K S^{-1}| \text{ and} \\ a_2 = |A||S^{-1}| + 2e^{|A|g}|B_K S^{-1}| - |B_K S^{-1}|. \end{array} \qquad (12)$$

Our final assumption is:

Assumption 3: There exists a constant $\delta \in (0, 1)$ such that

$$\begin{array}{l} \int_{t-2(\gamma_c+g)}^{t} \beta(q)\exp\left(-\int_{q}^{t} \alpha(\ell)\mathrm{d}\ell\right)\mathrm{d}q \\ \quad + \exp\left(-\int_{t-2(\gamma_c+g)}^{t} \alpha(\ell)\mathrm{d}\ell\right) \leq \delta \end{array} \qquad (13)$$

holds for all $t \geq 2(\gamma_c + g)$. □

We prove the following result:

Theorem 1: If Assumptions 1-3 hold, then the control

$$u(t) = K\left[e^{Ah(t)}x(t) + \int_{t-h(t)}^{t} e^{A(t-\ell)}Bu(\ell)\mathrm{d}\ell\right] \qquad (14)$$

renders the origin of (5) globally exponentially stable. □

Proof: In what follows, all (in)equalities are to be understood to hold for all $t \geq 0$, unless otherwise noted. We set $q(t) = |\dot{h}(t)|^2$ and

$$\begin{array}{l} \zeta(t) = e^{Ah(t)}x(t) + \Gamma(t),\ \text{where} \\ \Gamma(t) = \int_{t-h(t)}^{t} e^{A(t-\ell)}Bu(\ell)\mathrm{d}\ell\ . \end{array} \qquad (15)$$

Then the time derivative of ζ along all trajectories of (5) is

$$\begin{aligned} \dot{\zeta}(t) \ &= \ e^{Ah(t)}\dot{x}(t) + \dot{h}(t)Ae^{Ah(t)}x(t) + A\Gamma(t) \\ &\quad + Bu(t) - (1 - \dot{h}(t))e^{Ah(t)}Bu(t - h(t)) \\ &= \ [1 + \dot{h}(t)]A\zeta(t) + Bu(t) \\ &\quad -\dot{h}(t)A\int_{t-h(t)}^{t} e^{A(t-\ell)}Bu(\ell)\mathrm{d}\ell \\ &\quad + \dot{h}(t)e^{Ah(t)}Bu(t - h(t)) \\ &\quad + e^{Ah(t)}B[u(t-h(t)-\gamma(t)) - u(t-h(t))]. \end{aligned} \qquad (16)$$

Our control $u(t) = K\zeta(t)$ from (14) and Assumption 2 give

$$\dot{\zeta}(t) = R\zeta(t) + \dot{h}(t)A\zeta(t) + \omega(t,\zeta_t) + \kappa(t,\zeta_t) \quad (17)$$

where

$$\omega(t,\zeta_t) = -\dot{h}(t)A\Gamma(t) + \dot{h}(t)e^{Ah(t)}B_K\zeta(t-h(t)) \quad (18)$$

and

$$\kappa(t,\zeta_t) = e^{Ah(t)}B_K[\zeta(t-h(t)-\gamma(t)) - \zeta(t-h(t))] \, . \quad (19)$$

Then, since our choice of M in (9) gives $SMS = QA$, it follows from (8) that the time derivative of

$$V(\zeta) = \zeta^\top Q\zeta = |S\zeta|^2 \quad (20)$$

along all trajectories of (17) satisfies

$$\begin{aligned} \dot{V}(t) &\leq -cV(\zeta(t)) + 2\dot{h}(t)(\zeta(t)^\top S)M(S\zeta(t)) \\ &\quad + 2\zeta(t)^\top Q[\omega(t,\zeta_t) + \kappa(t,\zeta_t)] \\ &\leq -\alpha(t)V(\zeta(t)) + 2\sqrt{V(\zeta(t))}|S\omega(t,\zeta_t)| \\ &\quad + 2\sqrt{V(\zeta(t))}|S\kappa(t,\zeta_t)| \, . \end{aligned} \quad (21)$$

Next, note that our choice $q(t) = |\dot{h}(t)|^2$ gives

$$\begin{aligned} &|S\omega(t,\zeta_t)| \leq \sqrt{q(t)}\left|-N\int_{t-h(t)}^t e^{A(t-\ell)}B_K\zeta(\ell)\mathrm{d}\ell \right. \\ &\left. + Se^{Ah(t)}B_K\zeta(t-h(t))\right| . \end{aligned} \quad (22)$$

By replacing B_K by $B_KS^{-1}S$ in (22), it follows from our choice (20) of V that

$$\begin{aligned} &|S\omega(t,\zeta_t)| \leq \sqrt{q(t)}\Big[|Se^{Ah(t)}B_KS^{-1}|\sqrt{V(\zeta(t-h(t)))} \\ &+|N|\int_{t-h(t)}^t e^{|A|(t-\ell)}|B_KS^{-1}|\sqrt{V(\zeta(\ell))}\mathrm{d}\ell\Big] . \end{aligned}$$

Using the fact that $|N|/|A| \leq |S|$ when $A \neq 0$ and our bound g on $|h(t)|$ from (6), it follows that

$$\begin{aligned} &|S\omega(t,\zeta_t)| \leq \\ &2\sqrt{q(t)}|S||B_KS^{-1}|e^{|A|h(t)}\sup_{s\in[t-g,t]}\sqrt{V(\zeta(s))} \, . \end{aligned} \quad (23)$$

Combining (21) with (23), we get

$$\begin{aligned} &\dot{V}(t) \leq -\alpha(t)V(\zeta(t)) + 2\sqrt{V(\zeta(t))}|S\kappa(t,\zeta_t)| \\ &+ 4|S||B_KS^{-1}|\sqrt{q(t)}e^{|A|h(t)}\sup_{s\in[t-g,t]}V(\zeta(s)). \end{aligned} \quad (24)$$

Next, observe that since $Q = SS$, we can use the bound

$$\begin{aligned} |\dot{\zeta}(t)| \leq\ & |(R+\dot{h}(t)A)S^{-1}S\zeta(t)| \\ & + \Big|\dot{h}(t)A\int_{t-h(t)}^t e^{A(t-\ell)}B_KS^{-1}S\zeta(\ell)\mathrm{d}\ell \\ & - \dot{h}(t)e^{Ah(t)}B_KS^{-1}S\zeta(t-h(t))\Big| \\ & +\Big|e^{Ah(t)}B_K[S^{-1}S\zeta(t-h(t)-\gamma(t)) \\ & \quad - S^{-1}S\zeta(t-h(t))]\Big| \end{aligned} \quad (25)$$

and our formula (20) to get

$$\begin{aligned} |\dot{\zeta}(t)| \leq\ & |(R+\dot{h}(t)A)S^{-1}|\sqrt{V(\zeta(t))} + \sqrt{q(t)}|A| \\ & \times\int_{t-h(t)}^t \left|e^{A(t-\ell)}B_KS^{-1}\right|\sqrt{V(\zeta(\ell))}\mathrm{d}\ell \\ & +\sqrt{q(t)}\left|e^{Ah(t)}B_KS^{-1}\right|\sqrt{V(\zeta(t-h(t)))} \\ & +|e^{Ah(t)}B_KS^{-1}|\sqrt{V(\zeta(t-h(t)-\gamma(t)))} \\ & + |e^{Ah(t)}B_KS^{-1}|\sqrt{V(\zeta(t-h(t)))}. \end{aligned} \quad (26)$$

As an immediate consequence, we get

$$\begin{aligned} &|\dot{\zeta}(t)| \leq |(R+\dot{h}(t)A)S^{-1}|\sqrt{V(\zeta(t))} \\ &+ \sqrt{q(t)}|A||B_KS^{-1}|\int_{t-h(t)}^t \left|e^{A(t-\ell)}\right|\sqrt{V(\zeta(\ell))}\mathrm{d}\ell \\ &+\left|e^{Ah(t)}B_KS^{-1}\right|\Big[\left(\sqrt{q(t)}+1\right)\sqrt{V(\zeta(t-h(t)))} \\ &+\sqrt{V(\zeta(t-h(t)-\gamma(t)))}\Big] \\ &\leq \eta(t)\sup_{s\in[t-g-\gamma_c,t]}\sqrt{V(\zeta(s))}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} &\eta(t) = |(R+\dot{h}(t)A)S^{-1}| + 2e^{|A|h(t)}|B_KS^{-1}| \\ &+ 2\sqrt{q(t)}e^{|A|h(t)}|B_KS^{-1}| - \sqrt{q(t)}|B_KS^{-1}| \, . \end{aligned} \quad (28)$$

It readily follows from (12) and (28) that

$$|\dot{\zeta}(t)| \leq \mu(t)\sup_{s\in[t-g-\gamma_c,t]}\sqrt{V(\zeta(s))}, \quad (29)$$

where $\mu(t) = a_1 + a_2\sqrt{q(t)}$. Next, notice that

$$|\kappa(t,\zeta_t)| \leq |e^{Ah(t)}B_K|\int_{t-h(t)-\gamma(t)}^{t-h(t)}|\dot{\zeta}(\ell)|\mathrm{d}\ell \, . \quad (30)$$

Consequently,

$$\begin{aligned} &|\kappa(t,\zeta_t)| \leq |e^{Ah(t)}B_K|\int_{t-h(t)-\gamma(t)}^{t-h(t)}\mu(\ell)\mathrm{d}\ell \\ &\times\sup_{s\in[t-2g-2\gamma_c,t-h(t)]}\sqrt{V(\zeta(s))} \, . \end{aligned} \quad (31)$$

From (24), we deduce that

$$\begin{aligned} \dot{V}(t) &\leq -\alpha(t)V(\zeta(t)) \\ &\quad + 4|S||B_KS^{-1}|\sqrt{q(t)}e^{|A|h(t)} \\ &\quad\ \times\sup_{s\in[t-g,t]}V(\zeta(s)) \\ &\quad +2|S||e^{Ah(t)}B_K|\int_{t-h(t)-\gamma(t)}^{t-h(t)}\mu(\ell)\mathrm{d}\ell \\ &\quad\ \times\sup_{s\in[t-2g-2\gamma_c,t]}V(\zeta(s)) \\ &\leq -\alpha(t)V(\zeta(t)) \\ &\quad + \beta(t)\sup_{s\in[t-2g-2\gamma_c,t]}V(\zeta(s)). \end{aligned} \quad (32)$$

By Lemma 1 and Assumption 3, we conclude that $V(\zeta(t))$ converges exponentially to zero. Since Q is positive definite, it follows that $\zeta(t)$ converges exponentially to zero. Hence, since $h(t)$ is bounded, $x(t) = e^{-Ah(t)}(\zeta(t)-\Gamma(t))$ converges exponentially to zero, which proves the theorem. ■

IV. Alternative Approach

We next provide an approach for systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t-h(t)) + \varepsilon(t) \quad (33)$$

having state space $\mathbb{R}^n$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, ε is an unknown piecewise continuous disturbance, the (nonnegative) delay $h(t)$ is known, and:

Assumption 4: The pair (A,B) is controllable, h is C^1, and there is a constant $g > 0$ such that $\sup_t h(t) \leq g$. □

The stabilizability of (A,B) implies that we can construct matrices $A_f \in \mathbb{R}^{m\times m}$ and $B_f \in \mathbb{R}^{m\times n}$ such that

$$H = \begin{bmatrix} A & B \\ B_f & A_f \end{bmatrix} \quad (34)$$

is Hurwitz; see [31] for one such construction. We can also determine a symmetric and positive definite matrix Q and a constant $c > 0$ such that the inequality $QH + H^\top Q \leq -cQ$ holds. For each $\zeta = (\rho, \beta) \in \mathbb{R}^n \times \mathbb{R}^m$, we redefine V by

$$V(\zeta) = \zeta^\top Q \zeta \tag{35}$$

and we choose any constants $c_\rho > 0$ and $c_\beta > 0$ such that

$$|\beta| \leq c_\beta \sqrt{V(\zeta)} \ \text{ and } \ |\rho| \leq c_\rho \sqrt{V(\zeta)} \tag{36}$$

hold for all $\zeta = (\rho, \beta) \in \mathbb{R}^n \times \mathbb{R}^m$. For instance, we can take c_β and c_ρ both equal to $1/\sqrt{\lambda_{\min}(Q)}$. Setting $S = \sqrt{Q}$ as before, and choosing the functions $a(t) = 0.9c$ and

$$\begin{aligned} b(t) \;&=\; 2|S||\dot{h}(t)| \Big[c_\rho |A| + c_\beta |A| \textstyle\int_0^{h(t)} \left| e^{A\ell} B \right| \mathrm{d}\ell \\ &\quad + c_\beta \left| e^{Ah(t)} B \right| \Big], \end{aligned} \tag{37}$$

our final assumption is as follows:

Assumption 5: With the preceding choices of $a(t)$ and $b(t)$, there exist constants $T > 0$ and $\delta \in (0, 1)$ such that

$$e^{-0.9cT} + \textstyle\int_{t-T}^{t} b(\ell) e^{-(t-\ell)0.9c} \mathrm{d}\ell \;\leq\; \delta \tag{38}$$

holds for all $t \geq T + g$. □

We can then prove:

Theorem 2: If Assumptions 4-5 hold, then (33) in closed loop with $u(t) = \beta(t)$, where β is any solution of

$$\begin{aligned} &\dot{\beta}(t) = \\ &A_f \beta(t) + B_f \Big[e^{Ah(t)} x(t) + \textstyle\int_{t-h(t)}^{t} e^{A(t-\ell)} B \beta(\ell) \mathrm{d}\ell \Big], \end{aligned} \tag{39}$$

is input-to-state stable with respect to the set of all piecewise continuous functions $\varepsilon : [0, \infty) \to \mathbb{R}^n$. □

Proof: Choose any piecewise continuous function $\varepsilon : [0, \infty) \to \mathbb{R}^n$ and let Assumptions 4-5 hold. By Lemma 1 and (38), the theorem will follow once we prove that

$$\begin{aligned} \dot{V}(t) \;&\leq\; -a(t) V(\zeta(t)) + b(t) \sup\nolimits_{\ell \in [t-g, t]} V(\zeta(\ell)) \\ &\quad + \tfrac{40}{c} |S|^2 e^{2|A|g} \left| \varepsilon(t) \right|^2 \end{aligned} \tag{40}$$

holds along all solutions of the closed loop system for all $t \geq T + g$. Let $\rho(t)$ denote the quantity in squared brackets on the right side of (39), so $\dot{\beta}(t) = A_f \beta(t) + B_f \rho(t)$. Then

$$\begin{aligned} \dot{\rho}(t) \;&=\; A\rho(t) + B\beta(t) + \dot{h}(t) A \big[\rho(t) \\ &\quad - \textstyle\int_{t-h(t)}^{t} e^{A(t-\ell)} B \beta(\ell) \mathrm{d}\ell \big] \\ &\quad + \dot{h}(t) e^{Ah(t)} B \beta(t - h(t)) + e^{Ah(t)} \varepsilon(t). \end{aligned} \tag{41}$$

Since $\zeta(t) = (\rho(t), \beta(t)) \in \mathbb{R}^{n+m}$ for all $t \geq 0$, we get

$$\dot{\zeta}(t) = H\zeta(t) + \dot{h}(t) \begin{pmatrix} \psi(t) \\ 0 \end{pmatrix} + \begin{pmatrix} e^{Ah(t)} \varepsilon(t) \\ 0 \end{pmatrix}, \tag{42}$$

where

$$\begin{aligned} \psi(t) \;&=\; A \Big[\rho(t) - \textstyle\int_{t-h(t)}^{t} e^{A(t-\ell)} B \beta(\ell) \mathrm{d}\ell \Big] \\ &\quad + e^{Ah(t)} B \beta(t - h(t)) \,. \end{aligned} \tag{43}$$

Hence, (35) satisfies the following along trajectories of (42):

$$\begin{aligned} \dot{V}(t) \;&\leq\; -cV(\zeta(t)) + 2\dot{h}(t) \zeta(t)^\top Q \begin{pmatrix} \psi(t) \\ 0 \end{pmatrix} \\ &\quad + 2\zeta(t)^\top Q \begin{pmatrix} e^{Ah(t)} \varepsilon(t) \\ 0 \end{pmatrix}. \end{aligned} \tag{44}$$

Using the fact that $Q = SS$, we obtain $|\zeta^\top(t) Q| \leq |S| \sqrt{V(\zeta(t))}$, so

$$\begin{aligned} \dot{V}(t) \;&\leq\; -cV(\zeta(t)) + 2|S||\dot{h}(t)| \sqrt{V(\zeta(t))} |\psi(t)| \\ &\quad + 2|S| \sqrt{V(\zeta(t))} \left| e^{Ah(t)} \varepsilon(t) \right|. \end{aligned} \tag{45}$$

Also, our conditions (36) on $c_\beta > 0$ and $c_\rho > 0$ give

$$\begin{aligned} &|\psi(t)| \leq \left| e^{Ah(t)} B \right| c_\beta \sqrt{V(\zeta(t - h(t)))} \\ &+ |A| \Big[c_\rho \sqrt{V(\zeta(t))} + \textstyle\int_{t-h(t)}^{t} \left| e^{A(t-\ell)} B \right| c_\beta \sqrt{V(\zeta(\ell))} \mathrm{d}\ell \Big]. \end{aligned}$$

Consequently,

$$\begin{aligned} |\psi(t)| &\leq \Big[c_\rho |A| + c_\beta |A| \textstyle\int_0^{h(t)} \left| e^{A\ell} B \right| \mathrm{d}\ell \\ &+ c_\beta \left| e^{Ah(t)} B \right| \Big] \sup_{\ell \in [t-g, t]} \sqrt{V(\zeta(\ell))}. \end{aligned} \tag{46}$$

Combining (45) with (46), we get

$$\begin{aligned} \dot{V}(t) \;&\leq\; -cV(\zeta(t)) + b(t) \sup\nolimits_{\ell \in [t-g, t]} V(\zeta(\ell)) \\ &\quad + 2|S| \sqrt{V(\zeta(t))} \left| e^{Ah(t)} \varepsilon(t) \right| \,. \end{aligned} \tag{47}$$

Also, by the triangle inequality $jk \leq 0.1j^2 + 10k^2$ with j and k taken to be the corresponding terms in curly braces in (48), and our bound g on $h(t)$ from Assumption 4, we get

$$\begin{aligned} &2|S| \sqrt{V(\zeta(t))} \left| e^{Ah(t)} \varepsilon(t) \right| \\ &\leq \Big\{ \sqrt{cV(\zeta(t))} \Big\} \Big\{ \tfrac{2}{\sqrt{c}} |S| e^{|A|g} |\varepsilon(t)| \Big\} \\ &\leq \tfrac{c}{10} V(\zeta(t)) + \tfrac{40}{c} |S|^2 e^{2|A|g} \left| \varepsilon(t) \right|^2 \end{aligned} \tag{48}$$

for all t. If we now use (48) to upper bound the last right side term in (47), the desired estimate (40) then follows from our choice of $a(t)$. This proves the theorem. ■

V. Illustration

We revisit the unstable second order linear dynamics from [7], which is the special case of the following when the unknown perturbation term $\varepsilon(t)$ is the zero function:

$$\dot{z}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t - h(t)) + \varepsilon(t). \tag{49}$$

In this section, we only apply Theorem 2 and therefore assume that $h \in C^1$ and bounded by some constant $g > 0$, but we can also apply Theorem 1 to cover cases where the delay can have discontinuities and be uncertain. Choosing

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \ \text{ and } \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{50}$$

the corresponding matrix $M = A + BK$ is Hurwitz with the choice $K = -(2, 3)$. In fact, we can satisfy $PM + M^\top P \leq -cP$ with the choices

$$P = \begin{pmatrix} \frac{4}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix} \ \text{ and } \ c = 0.735089. \tag{51}$$

Also, using the fact that $|B| = 1$ and relatively simple calculations, one checks that Assumptions 4-5 will hold if there are also constants $\delta \in (0,1)$ and $T > 0$ such that

$$\int_{t-T}^{t} |\dot{h}(\ell)| \mathrm{d}\ell \ \leq \ \frac{\delta - e^{-0.9cT}}{2|S|\left(c_\rho |A| + c_\beta e^{|A|g}(1+g|A|)\right)} \tag{52}$$

holds for all $t \geq 0$, where $S = \sqrt{Q}$, Q is defined by

$$Q = \left(\begin{array}{cc} P + \frac{\nu}{2} K^\top K & -\frac{\nu}{2} K^\top \\ -\frac{\nu}{2} K & \frac{\nu}{2} I_m \end{array} \right), \tag{53}$$

$\nu > 0$ is any constant, and the constants $c_\rho > 0$ and $c_\beta > 0$ satisfy (36) with $V(\zeta) = \zeta^\top Q \zeta$. This provides an interesting alternative to the corresponding result in [7] because it allows us to conclude input-to-state stability with respect to additive uncertainties ε (which were not considered in [7]) using the alternative controller from our Theorem 2 (which is based on a dynamic extension that was also not considered in [7]), and because we include the degree of freedom $\nu > 0$.

VI. Conclusions

We proposed two approaches for proving stability of linear time invariant systems that apply to cases where the delay could be unknown and not necessarily C^1, and where the system is subjected to additive uncertainties on the right side. In both cases, we allow the known C^1 part $h(t)$ of the delay to exhibit chattering phenomena, where $h(t)$ is bounded, but where there is a bound on an integral average of $|\dot{h}|$ instead of on $\dot{h}$ itself. Therefore, our work built on the significant work [7] on chattering delays, by also allowing uncertainties in the delay or in the system that were not considered in [7]. There are many degrees of freedom in applying our methods, such as the coefficient matrices in our dynamic extensions. In our future work, we hope to develop ways to exploit the degrees of freedom to make our work applicable to the broadest possible class of systems and chattering delays.

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